

# Shock waves and Birkhoff's theorem in Lovelock gravity

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Spherically symmetric shock waves are shown to exist in Lovelock gravity. They amount to a change of branch of the spherically symmetric solutions across a null hypersurface. The implications of their existence for the status of Birkhoff's theorem in the theory is discussed.

## I. INTRODUCTION

In general relativity Birkhoff's theorem is roughly the statement that outside a spherically symmetric (even time-varying) source the metric field is necessarily static and it is given by the Schwarzschild solution [1][2][3]. A better statement of the theorem is that the vacuum field equations imply that a spherically symmetric  $C^2$  solution is locally equivalent to a maximally extended Schwarzschild metric [4]<sup>1</sup>. An important refinement came with the work of Ref. [7], building on the work of Refs. [8][9]: Schwarzschild metric is the unique spherically symmetric family of vacuum solutions and Birkhoff's theorem holds even if we lower differentiability class to  $C^0$ . The result which encapsulates, one may say, the essence of what makes the theorem possible, is that spherically symmetric shock waves do not exist. In more words, there cannot be a vacuum solution respecting spherical symmetry whose first derivative becomes discontinuous across a null hypersurface. This result was obtained already in [9]. Had such a solution existed, any statement of Birkhoff's theorem for smooth metrics would be strongly weakened. On the other hand, the inexistence of the spherically symmetric shock waves may be attributed to the uniqueness of the spherically symmetric smooth metric of a given mass. There simply cannot be a non-trivial matching across a (null) hypersurface respecting the symmetry.

In dimensions higher than four, Einstein field equations can be formally thought of as a special case of the Lovelock field equations [10]. The additional terms are of higher order in curvature nonetheless they still are second order differential equations for the metric tensor. Originally, at least in the recent years, an interest in Lovelock gravity came through studies in low energy effective actions in string theory [11]. In the last decade the interest was revived due the higher dimensions becoming popular, essentially through works such as [12][13]. In the last few years a certain amount of interest in Lovelock gravity was further drawn, taking also an attractive turn. The shear viscosity/entropy density bounds for a conformal field theory living on the boundary of AdS [14][15] are modified and restricted when ap-

plying causality conditions on the conformal field theory with a Lovelock gravity dual in the bulk [16][17]. Moreover these restrictions coincide unexpectedly to ones obtained through positive energy conditions on scattering processes in super-conformal field theories [18]. These findings led to a series of new works, see e.g. [19–30] related to Lovelock gravity.

Establishing Birkhoff's theorem in Lovelock gravity requires a more careful phrasing than in general relativity. Perhaps the most complete presentation of the theorem has been given in Ref. [31] building on the work of [32]. Comments in that direction have been presented in other works [33–36]. The statement of the theorem is one of uniqueness: The vacuum spherically symmetric metrics of differentiability class  $C^2$  and for generic values of the couplings of the theory, are locally equivalent to a specific family of solutions, some of which are the Lovelock black hole metrics. A characteristic feature of Lovelock gravity is that these solutions are multi-valued.

Now, what about metrics of differentiability lower than  $C^2$ ? In Ref. [37] spherically symmetric *vacuum* solutions with  $C^0$  piecewise  $C^\infty$  metrics where explicitly constructed. Their metric is everywhere smooth except at certain time-like as well as space-like hypersurfaces where it is only continuous; its first derivative is discontinuous. That is, non-trivial *vacuum shells* with time- and space-like trajectories were explicitly shown to exist. These imply two things. One, there is a problem of non-causal evolution due to the spacelike vacuum discontinuities. Second, any staticity interpretation of a Birkhoff's theorem in this theory is very much weakened by the possibility of a series of spherical vacuum shells whizzing along spacetime. The existence of spherical shock waves in Lovelock gravity, that is of spherical vacuum shells moving along light-like trajectories, are the subject of the present work. They elegantly appear as a mere *change of branch* along a null hypersurface. Now even if one finds a way, by some nice principle, to censor out the non-null discontinuities, the null ones cannot be excluded. They are gravitational shock waves, perfectly existent in general relativity, with the exception of the spherical ones which gives rise to Birkhoff's theorem. In Lovelock gravity the spherical ones exist and appear in fact as fairly natural objects in the theory along with its multiple branch solutions. Presumably, all these non-trivial vacuum shell spherical symmetric solutions can be regarded as large (non-linear) scalar perturbations of a given smooth spherically symmetric metric.

In order to show that configurations such as the ones

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<sup>1</sup> Useful additional background on Birkhoff's theorem in Einstein gravity is provided in the Refs. [5][6].

discussed in the previous paragraph are indeed solutions of a given theory, requires to formulate and solve, one way or another, junction or matching conditions in that theory. In general relativity, the matching conditions for the time- or space-like hypersurface where given their final form in Ref. [38]. Shock waves and null hypersurface matching conditions have been studied in various old and more recent works [8][39–47]. The problem we set ourselves to solve is to construct certain shock wave solutions in Lovelock gravity. To do this we shall exploit the formulation of matching conditions first presented in Ref. [48] and further elaborated in [49] and [50]. The basic facts of the method can be summarized as follows. The gravitational field is described by the vielbein and the spin connection. Consider a sequence of smooth configurations which are arbitrarily close approximations of a given discontinuous one, e.g. one where the spin connection becomes discontinuous across hypersurfaces [which may also intersect]. The hypersurfaces divide spacetime up into ‘bulk’ regions where fields are smooth. At the hypersurfaces the discontinuous fields are ill-defined. Then the action functional of the theory evaluated on that sequence can be shown to converge to a new action functional involving only well defined information: fields outside the hypersurfaces. The fields in the vicinity of each side of the hypersurface contain all the information about the discontinuity, therefore nothing more needs to appear in the equations. [That presumably means that no explicit reference to the induced fields on the hypersurface is required.] Working this way the hypersurfaces need only be locally smooth; their causal character, null or non-null, is irrelevant. The whole analysis is done off-shell, therefore the equations of motion for the discontinuous fields obtained by Euler-Lagrange variation of the new action are the well defined limit of the usual equations of motion for the smooth fields. The new action contains explicitly terms for each hypersurface. The matching conditions are the parts of the equations of motion with support at the hypersurfaces. The convergence of the off-shell action to the new action makes them well defined: the bulk fields on the sides of a hypersurface of *any* smooth configuration approximating well the discontinuity must obey those matching conditions.

The way of ‘matching’ in the formulation of [48–50] is somewhat different than usual: One works with globally defined fields with a certain amount of discontinuity at some places and attempts to write down an action functional for these fields which is well defined in the sense stated above. The spirit of the analysis reflects nicely the old work of Papapetrou and Treded [8] in general relativity we already mentioned. The principle is that field configurations of differentiability lower than the order of the field equations, are also solutions of the theory if they can be approximated arbitrarily well by sequences of smooth *solutions*. A subtlety here is that such sequences is not always easy to make explicit. One such case is the vacuum solutions we are interested in. On the

other hand the well-defined-ness of the matching condition is established by sequences of smooth configurations off-shell. There are always non-vacuum smooth configurations approximating arbitrarily well a given vacuum discontinuous configuration. In that limiting sense solutions of the matching conditions may be regarded as meaningful stationary point of the classical action.

As the formulation of the matching conditions we shall use is not a well known one, we will devote the section III in explaining the technical and physical details leading to the matching condition, equation (15). In section IV the formulation is suitably applied to obtain the alleged shock wave solution, and certain additional comments are given in the final section.

## II. LOVELOCK GRAVITY

To fix ideas consider a specific example of Lovelock gravity. Let our theory be Einstein gravity with cosmological constant  $\lambda$  supplemented by the quadratic Lovelock term in five dimensions. This is usually termed as Einstein-Gauss-Bonnet gravity. We denote the coupling of the quadratic Lovelock term by  $\alpha$ . It has length dimension  $L^{+2}$ . The Lagrangian of the theory reads

$$S = -\lambda \int_M \sqrt{-g} + \frac{1}{2\kappa^2} \int_M \sqrt{-g} \{R + \alpha(R^2 - 4R_{\alpha\beta}R^{\alpha\beta} + R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta})\}. \quad (1)$$

The spherically symmetric  $C^2$  solutions of the theory (1) are the Boulware-Deser metrics [51–54] (their general forms in Lovelock gravity where first studied in [55]):

$$ds^2 = -f_{BD}dt^2 + f_{BD}^{-1}dr^2 + r^2d\Omega^2, \quad (2)$$

$$f_{BD} = 1 + \frac{r^2}{4\alpha} \left\{ 1 \pm \sqrt{1 + \frac{4\alpha\kappa^2\lambda}{3} + \frac{16\alpha M}{r^4}} \right\}.$$

$d\Omega^2$  is a metric of the unit round 3-sphere.  $M$  is a constant of integration related to the mass of the solution. The Boulware-Deser metrics *are not single-valued*: they have two branches, corresponding to the  $\pm$  sign choice, which have very different properties. For one thing, there are no black hole solutions in the metrics of the branch corresponding to the sign  $+$ . Also for  $\lambda = 0$  this branch of the solution is not asymptotically flat: it is asymptotically anti-de Sitter for  $\alpha > 0$  and de Sitter for  $\alpha < 0$ . A related fact is that only the metrics of the branch  $-$  converge to Einstein gravity metrics in the limit  $\alpha \rightarrow 0$ . It has been argued that the asymptotic vacuum of the branch  $+$  metrics is unstable due the ghost excitations [51][56]. On this basis one may discard it from the beginning, but that is rather too quick. These metrics are an inherent property of Lovelock gravity and their effects should be studied before they are discarded with reason, or not discarded at all. For one thing, their presence might make things better or worse, stability-

or other-wise, for the well behaved branch metrics; both situations are of physical importance. One should bear in mind that the two types of metrics are naturally intermingled by the theory. Lovelock gravity allows  $C^0$  piecewise smooth non-trivial vacuum configurations to be cut-and-paste constructed out of metrics of the same or different branch as shown in Ref. [37], see also [57]. New such configurations associated with shock waves are presented here.

### III. THE LAGRANGIAN AND EQUATIONS OF MOTION

#### A. First order formalism

Proceeding with the mathematical analysis, it is far more convenient than working with the metric tensor  $g_{\mu\nu}$  directly to use the vielbein  $E_\mu^a$  and an  $SO(4,1)$  spin connection  $\omega_\mu^{ab}$  one-forms over  $M$  as our variables. They will be treated as differential forms  $E^a \equiv dx^\mu E_\mu^a$ ,  $\omega^{ab} \equiv dx^\mu \omega_\mu^{ab}$ . We will do our differentiations and integrations using exterior calculus<sup>2</sup>. The metric tensor  $\eta_{ab} = (- + \dots +)$  and the volume anti-symmetric tensor (form)  $\epsilon_{abcde}$  are the invariant tensors  $SO(4,1)$ . From them we can build invariant forms over the spacetime manifold.

We will use a convenient notation for contraction with the volume form, for example

$$\epsilon(\Omega E^3) := \epsilon_{abcde} \Omega^{ab} \wedge E^c \wedge E^d \wedge E^e.$$

Its convenience can be seen in variations and differentiations, e.g.  $\delta\{\epsilon(\Omega E^3)\} = \epsilon(\delta\Omega E^3) + 3\epsilon(\Omega E^2 \delta E)$ . The invariance of the contraction also implies that

$$d(\epsilon(\dots)) = \epsilon(D(\dots)),$$

where  $D$  is the covariant derivative associated with the spin connection. From now on the wedge symbol  $\wedge$  will be dropped in the wedge product of forms.

The spin connection throughout this work will be Levi-Civita i.e. torsion-free:  $T^a := DE^a \equiv dE^a + \omega_b^a E^b = 0$ .  $d$  is the exterior calculus derivative operator. Its nilpotence,  $dd = 0$ , holds on all  $C^2$  forms. Then the affine connection on the tangent bundle is Levi-Civita and defines the usual covariant derivative of general relativity. The spin connection is not a tensor. The curvature  $\Omega_b^a := d\omega_b^a + \omega_c^a \omega_b^c$  is an  $SO(4,1)$  tensor and a two-form over  $M$ . We can write that more compactly treating the forms  $\omega$  and  $\Omega$  as matrices:  $\Omega = d\omega + \omega^2$ . The curvature satisfies the Bianchi identity:  $D\Omega \equiv d\Omega + \omega\Omega - \Omega\omega = 0$  identically, as one may verify.

The curvature form is related to Riemann tensor by the relation  $\Omega^{ab} = \frac{1}{2} E_\mu^a E_\nu^b R^{\mu\nu}{}_{\rho\sigma} dx^\rho dx^\sigma$ . In other

words to obtain the Riemann tensor form the curvature form one has to invert the matrix  $E_\mu^a$ . We shall not need to do that as we will work exclusively with forms, though invertibility will not be evaded anywhere. Thus the vielbein and connection formulations is equivalent to the metric formulations. Then the action (1) can be written in the following form

$$S = -\frac{1}{2}c_0 \int_M \epsilon(E^5) + \frac{1}{2}c_1 \int_M \epsilon(\Omega E^3) + \frac{1}{2}c_2 \int_M \epsilon(\Omega \Omega E). \quad (3)$$

The coupling constants  $c_0$ ,  $c_1$  and  $c_2$  are related to the more usual couplings by

$$c_0 \equiv \frac{\lambda}{60}, \quad c_1 \equiv \frac{1}{3!\kappa^2}, \quad c_2 \equiv \frac{\alpha}{\kappa^2}, \quad (4)$$

and  $\kappa^2 = 8\pi G$ , where we introduce  $G$  as the Newton's constant (though conventions differ in the literature). The cosmological constant appears usually as  $\Lambda \equiv \kappa^2 \lambda$ . We shall prefer using mostly the  $c$  couplings. The length dimension of  $c_1$  is  $L^{-3}$  and the dimension of  $c_2$  is  $L^{-1}$ .

#### B. Manufacturing a discontinuity

Imagine then spacetime consisting of three regions: two open bulk regions  $N$ ,  $\bar{N}$  and a closed region  $\Delta$  surrounding the hypersurface  $\Sigma$  we would like to be locus of the discontinuity. The action may written as

$$S = \int_N \mathcal{L} + \int_\Delta \mathcal{L} + \int_{\bar{N}} \mathcal{L},$$

where  $\mathcal{L}$  is a Lovelock gravity Lagrangian.

Consider the limit  $\Delta \rightarrow \Sigma$ . It is not that this part of spacetime actually shrinks but rather the values of the fields  $E$  and  $\omega$  are moved. This can be done by choosing a sequence of configurations imitating this process. On the common boundary of  $\Delta$  with each of  $N$  and  $\bar{N}$  the values of the fields  $E$  and  $\omega$  are held fixed (and in general different). These values become the values of the fields on each side of  $\Sigma$  as it is embedded in each of  $N$  and  $\bar{N}$ . These are *bulk* fields and they are well defined as long as there is a coordinate neighborhood covering the hypersurface  $\Sigma$ .

The limit we described here as  $\Delta \rightarrow \Sigma$ , or any limit equivalent to it, was studied for any theory built out of forms fields alone, such as  $E$  and  $\omega$ , in Ref. [50]. The general result is as follows. Evaluate the action  $S$  on an arbitrary sequence of configurations consistent with the limit  $\Delta \rightarrow \Sigma$  as described above. Then for each such sequence of configurations one obtains a sequence of functionals, whose convergence needs to be checked. Convergence depends on the amount of discontinuity allowed at  $\Sigma$ . That should be formalized into specific conditions on the fields, the 'continuity conditions'. Choosing them appropriately the action *converges* to a new action, which necessarily depends only on well defined quantities, the

<sup>2</sup> Convenient references for this formulation are [58] and [59].

bulk fields. [That convergence we usually describe as ‘well-defined-ness’ of the new action.] Such continuity conditions are not unique and depend on the theory; in general they can be quite strange. In the limit  $\Delta \rightarrow \Sigma$  the action necessarily involves distributional fields; on the other hand the new action depends only on the fixed bulk fields on the sides of  $\Sigma$ . Everything is done off-shell, therefore whatever holds for the actions it holds also for the equations of motion. The convergence of the action to the new action means then two things. First, the equations of motion of the distributional fields are meaningful, and secondly, they have been neatly disentangled, the necessary integrations having been essentially done, into the form of the equations of motion deriving from the new action. One may then say that the new action is an equivalent form of the action of the theory for discontinuous fields.

A straightforward application of these ideas is offered by Lovelock gravity under the most usual continuity condition. In [50] it was shown that that one may write down a new action for Lovelock gravity in the presence of discontinuities if the vielbein  $E$  is continuous and the spin connection  $\omega$  is discontinuous, a result essentially contained also in [49]. Lorentz transformations are still a local symmetry of the new action [48][49]. Therefore our continuity condition explicitly is: The vielbein is continuous across the hypersurface modulo a Lorentz transformation. This continuity condition will be applied and explained in more detail in section IV.

Based on these results we may write the action (3) in the presence of the discontinuity at  $\Sigma$  in the equivalent form:

$$\begin{aligned} S = & -\frac{1}{2}c_0 \int \epsilon(E^5) \\ & + \frac{1}{2}c_1 \int \epsilon(\Omega E^3) + \frac{1}{2}c_1 \int_{\Sigma} \epsilon((\bar{\omega} - \omega)E^3) \\ & + \frac{1}{2}c_2 \int \epsilon(\Omega \Omega E) \\ & + \frac{1}{2}c_2 \int_{\Sigma} \epsilon((\bar{\omega} - \omega)\{(\bar{\Omega} + \Omega) - \frac{1}{3}(\bar{\omega} - \omega)^2\}E). \end{aligned} \quad (5)$$

The bulk integrals are over  $N \cup \bar{N}$ . Writing  $(\bar{\omega} - \omega)^2$  matrix multiplication of the forms is understood.  $\omega$  and  $\Omega$  are respectively the spin connection and its curvature on  $N$  evaluated at  $\Sigma$ ; more precisely, as  $N$  is open, they are the limits of these variables as we approach  $\Sigma$  from within  $N$ . Similarly for  $\bar{\omega}$  and  $\bar{\Omega}$  in  $\bar{N}$ . The vielbein is continuous modulo a Lorentz transformation therefore we can express everything in terms of a single vielbein field  $E$ .

### C. Equations of motion

Equations of motion are obtained by varying the action with respect to the fields,  $E$  and  $\omega$ .

It was shown in [48] and more elegantly in [49] that the variation of the action w.r.t. the spin connection  $\omega$  trivially vanishes upon imposing that torsion is zero i.e. that the connection provides the familiar Levi-Civita covariant derivative and (3) is indeed equivalent to the Lovelock gravity action (1). That is, all equations of motion, including the matching conditions, are obtained by varying w.r.t. the vielbein  $E$ . This is a merely algebraic variation of the Lagrangian.

Explicitly, equations of motion are obtained for every smooth submanifold in the problem, that is, the bulk regions, the hypersurfaces, and in the general case the intersections of the hypersurfaces. There is a term in the Lagrangian for each one of these submanifolds. Let  $\delta E^a = \lambda(x)^a_b E^b$  be the variation of the vielbein, where  $\lambda(x)^a_b$  is an arbitrary smooth field with support in any one of the submanifolds. The equations of motion for that submanifold are obtained from the corresponding term in the Lagrangian. From the bulk regions we obtain the usual smooth field equations of motion; from the hypersurfaces and their intersection we obtain a set of matching conditions. If in general there are other fields with energy tensor  $T^a_b$ , the field equations of each submanifold read

$$\delta \mathcal{L} = T^a_b \lambda(x)^b_a \quad (6)$$

times its volume element.

In vacuum,  $T^a_b = 0$ . That is,

$$\delta \mathcal{L} = 0. \quad (7)$$

In vacuum there is no need to identify components; one simply sets to zero all non-trivial terms in this equation.

Here, we have the bulk regions  $N$ ,  $\bar{N}$  and the hypersurface  $\Sigma$  to deal with.

The field equations in the region  $N$  read

$$-5c_0 \epsilon(E^4 \delta E) + 3c_1 \epsilon(\Omega E^2 \delta E) + c_2 \epsilon(\Omega \Omega \delta E) = 0. \quad (8)$$

A similar expression holds in  $\bar{N}$  for the barred fields.

Consider the spherically symmetric metric

$$ds^2 = -g^2 dt^2 + \frac{dr^2}{g^2} + r^2 d\Omega^2, \quad (9)$$

where  $g = g(r)$  and  $d\Omega^2$  is the metric of the unit round 3-sphere. The metric (9) can be written in terms of the vielbein one-forms

$$E^0 = g dt, \quad E^1 = \frac{dr}{g}, \quad E^i = r \tilde{E}^i. \quad (10)$$

$\tilde{E}^i$  is a vielbein of the unit round 3-sphere. We denote by  $\tilde{\omega}^{ij}$  its Levi-Civita connection. The curvature of  $\tilde{\omega}^{ij}$  necessarily is  $\tilde{\Omega}^{ij} = \tilde{E}^i \tilde{E}^j$ . The Levi-Civita connection  $\omega^{ab}$  of  $E^a$  reads

$$\omega^{01} = g' g dt, \quad \omega^{i1} = g \tilde{E}^i, \quad \omega^{ij} = \tilde{\omega}^{ij}, \quad (11)$$

The curvature  $\Omega^{ab}$  of the connection  $\omega^{ab}$  reads

$$\begin{aligned}\Omega^{01} &= -\frac{(g^2)''}{2} E^0 E^1, \\ \Omega^{0i} &= -\frac{(g^2)'}{2r} E^0 E^i, \quad \Omega^{1i} = -\frac{(g^2)'}{2r} E^1 E^i, \\ \Omega^{ij} &= \frac{1-g^2}{r^2} E^i E^j.\end{aligned}\tag{12}$$

It is then straightforward to show that the entire content of (8) amounts to a single differential equation linear in  $g^2 - 1$ . Its solution is

$$g^2 - 1 = \frac{3c_1}{2c_2} r^2 \left\{ 1 \pm \sqrt{1 + \frac{c_2 \lambda}{27c_1^2} + \frac{C}{r^4}} \right\}.\tag{13}$$

Translating the  $c$  couplings into the more usual ones via (4) we indeed obtain the Boulware-Deser metric (2) with  $g^2 = f_{BD}$ . The integration constant  $C$  is of course related to the mass of the solution.

For future use let us note the following. Denote by  $g_{BD\pm}$  the function  $g$  for the respective branch of the Boulware-Deser solution. One observes that

$$g_{BD+}^2 + g_{BD-}^2 = 2 + \frac{3c_1}{c_2} r^2.\tag{14}$$

This relation will be useful later on.

The vacuum field equation  $\delta\mathcal{L} = 0$  for  $\Sigma$  is the matching condition

$$\begin{aligned}i_\Sigma^* \left\{ \frac{3}{2} c_1 \epsilon((\bar{\omega} - \omega) E^2 \delta E) + \right. \\ \left. + \frac{1}{2} c_2 \epsilon((\bar{\omega} - \omega) \{\bar{\Omega} + \Omega - \frac{1}{3}(\bar{\omega} - \omega)^2\} \delta E) \right\} = 0.\end{aligned}\tag{15}$$

$i_\Sigma^*$  denotes the pull-back of the form into  $\Sigma$ .

Equation (15) is derived under the hospitable assumption that the vielbein in spacetime is continuous across  $\Sigma$  modulo a Lorentz transformation.

When  $\Sigma$  is time- or space-like, there is a unique geometry on  $\Sigma$  induced from the bulk. One may choose a vielbein adapted to  $\Sigma$  [all but one its components are tangentialy oriented]. Pulled back into  $\Sigma$  defines an induced vielbein, which is regarded as intrinsic to  $\Sigma$ . Solving the zero torsion condition one obtains a unique Levi-Civita spin connection for that vielbein. This is because projecting  $\eta_{ab}$  into those tangential directions leaves us with an non-degenerate (invertible) tensor. The obtained connection coincides with the tangential components of  $\omega$ . The induced fields from  $N$  and  $\bar{N}$  may differ only by a Lorentz transformation. That is there is indeed a unique geometry on the hypersurface inherited from the bulk, and is regarded as intrinsic to it. Equation (15), which in general involves a ‘matter’ energy tensor  $T_b^a$  on its r.h.s. according to (6), it is equivalent to the matching conditions first written down in [60][61]. In those works the matching conditions appeared in the usual kind of formulation (as set by the work of Israel [38]) which involves explicitly the intrinsic geometry of  $\Sigma$  and the extrinsic

curvatures of  $\Sigma$  w.r.t. the bulk regions  $N$  and  $\bar{N}$ . The equivalence of that formulation to (15) was shown in [48] and re-visited in [57].

When  $\Sigma$  is null, the induced vielbein may also be regarded as an intrinsic vielbein; it does span the space of tangential one-forms. The difference is that one direction is null and the projection of  $\eta_{ab}$  into the tangential direction is degenerate (non-invertible). There is no unique Levi-Civita connection for the induced vielbein, thus no unique inherited geometry. Penrose [41], see also [45], classifies (three) different induced geometries which can be regarded as intrinsic to the hypersurface; these geometries differ on the level the induced notions of parallel transport along  $\Sigma$  agree. These notions, presumably, depend on the derivatives of the metric. All this wealth of structure available in the null hypersurface is reflected in practice as follows. The normal vector is null therefore orthogonal to itself [and indeed tangential on the hypersurface]. Thus one cannot construct projections along the tangential directions of  $\Sigma$  just by knowing its normal vector. That, combined with the non-uniqueness of the intrinsic geometry, implies that splitting the fields into components containing information intrinsic and extrinsic to  $\Sigma$  is less straightforward in the null case. Formulations of the matching conditions involving explicitly the intrinsic geometry of the hypersurface necessarily have qualitative differences in the null and non-null case. Unifying descriptions of the two cases can be found [45][47] but they rather emphasize the peculiarities of the null case.

In our formulation of the matching conditions, leading to (15), only *bulk field* information in the vicinity of  $\Sigma$  is involved. The intrinsic geometry information exists implicitly in the continuity conditions on the fields. That is, it exists in a minimal choice [so that to accommodate as many cases as cases as possible] of which field variables are continuous across  $\Sigma$ ; the rest are discontinuous, as restricted of course by the field equations. Thus the causal character of  $\Sigma$ , time- or space-like, or null, enters the matching conditions only through our requirements on the fields in each specific problem. Formula (15) holds in all cases.

In a most general setting, studied in Ref. [50], any chosen continuity conditions is such that the matching conditions are well defined, in the sense explained previously. On this basis, the acceptable continuity conditions are certainly not unique [50]. In practice, one chooses acceptable continuity conditions which are geometrically intelligible. As we have already mentioned, our choice here is a usual one: We have required that the vielbein in spacetime is continuous across  $\Sigma$  modulo a Lorentz transformation, which means that the metric tensor  $g = \eta_{ab} E^a \otimes E^b$  in spacetime is continuous.

It is worth to mention that the choice of the vielbein  $E^a$  and the connection  $\omega_b^a$  as gravitational field variables plays an important role. A choice of variables involving, for example, the metric tensor would necessarily involve its inverse. In any such case our formulation

of the problem in terms of globally defined discontinuous fields would become complicated and almost none of the tools used in Refs. [48–50] could be applied effectively in order to derive equations such as (15) and prove facts about them. Moreover, the one-forms  $E^a$  and  $\omega_b^a$  are tensor valued in the Lorentz group. They can be written w.r.t. to arbitrary coordinates in *each* region in spacetime. The continuity condition on the vielbein across their common boundaries i.e. the hypersurfaces, is that the vielbein is continuous modulo Lorentz transformations, a local symmetry of the action (5). Indeed, in every specific problem, one can construct a Lorentz transformation between the vielbeins on the sides of a hypersurface. This transformation is then used in (15) in order to express all fields in it w.r.t. the same basis; this completes the statement of the matching conditions. Therefore there is no need to explicitly use special coordinates chosen such that continuity is ensured. In this sense the formulation employed here parallels the work in Ref. [47], though the freedom to use any convenient coordinates on each side of the hypersurface is achieved quite differently.

In all, one has in hand a formulation of the matching conditions which can be uniformly applied to the space- and time-like as well as to the null hypersurface which is the case of interest. Of course writing down the correct equations of motion is different than understanding their content. In the null case the wealth of structure is revealed only after systematic general analysis. A full discussion of the null hypersurface matching conditions is far beyond the scope of the present work. In what follows we apply (15) to show that the shock waves we advertised at the beginning of our work indeed exist.

#### IV. SHOCK WAVES

Continuity of the vielbein in spacetime  $M$  means the following. Each vielbein field,  $E$  and  $\bar{E}$ , can be extended from the region it is defined,  $N$  and  $\bar{N}$  respectively, across  $\Sigma$  and into a neighborhood of the other region. In the overlap  $E$  and  $\bar{E}$  may differ by a Lorentz transformation which is a local symmetry of action (5). Let us put this condition into formulas for the problem of interest.

On the manifolds  $N$  and  $\bar{N}$  the vielbein is given respectively by

$$E^0 = g dt, \quad E^1 = \frac{dr}{g}, \quad E^i = r \tilde{E}^i \quad (16)$$

and

$$\bar{E}^0 = \bar{g} d\bar{t}, \quad \bar{E}^1 = \frac{d\bar{r}}{\bar{g}}, \quad \bar{E}^i = \bar{r} \tilde{E}^i. \quad (17)$$

Recall that  $\tilde{E}^i$  is a vielbein on the unit round 3-sphere introduced in the previous section, formulas (9) and (10).

Let  $\Sigma$  be a hypersurface along which

$$\frac{dr}{g} - g dt = 0, \quad \frac{d\bar{r}}{\bar{g}} - \bar{g} d\bar{t} = 0, \quad (18)$$

in  $N$  and  $\bar{N}$  respectively. More precisely, the pull-back of the forms on the l.h.s. of these relations into  $\Sigma$  vanishes.  $\Sigma$  is null.

A null hypersurface in spacetime dimension five is generated by a three-parameter family of null geodesics, one through each point of the hypersurface. The null geodesics along  $\Sigma$  satisfy (18), with parameters the points on the unit 3-sphere. Along these geodesics  $ds^2 = 0$  thus the induced metric is degenerate. The degenerate induced metric is another way, especially when working directly with the metric tensor, to recognize a null hypersurface.

The field  $E^a$  is assumed extendible across  $\Sigma$  and into the manifold  $\bar{N}$ . Similarly the field  $\bar{E}^{\bar{a}}$  is assumed extendible across  $\Sigma$  and into the manifold  $N$ . Explicitly that means the following. There is a neighborhood of  $M$  around  $\Sigma$  where  $E^a$  and  $\bar{E}^{\bar{a}}$  co-exist. This is the overlap of their support in spacetime. In that neighborhood define fields  $\bar{E}^a$  and  $\bar{E}^{\bar{a}}$  such that

$$\bar{E}^a \text{ is a rotated } \bar{E}^{\bar{a}} \text{ so that } \bar{E}^a|_{\Sigma} = E^a|_{\Sigma}, \quad (19)$$

$$E^{\bar{a}} \text{ is a rotated } E^a \text{ so that } E^{\bar{a}}|_{\Sigma} = \bar{E}^{\bar{a}}|_{\Sigma},$$

By ‘rotated’ we of course mean Lorentz transformed

$$\bar{E}^a = \Lambda^a_{\bar{a}} \bar{E}^{\bar{a}}, \quad (20)$$

everywhere in the overlap of  $E^a$  and  $\bar{E}^{\bar{a}}$  in  $M$ . The inverse of the matrix  $(\Lambda^a_{\bar{a}})$  is the matrix  $(\Lambda^{\bar{a}}_a)$ . Therefore  $\bar{E}^{\bar{a}} = \Lambda^{\bar{a}}_a \bar{E}^a$  in that neighborhood. Completely analogous formulas hold for the unbarred fields as we extend  $E^a$  into  $\bar{N}$ . We shall not need explicitly these formulas.

Continuity of the  $a = i$  components of the vielbein implies that  $r = \bar{r}$  along  $\Sigma$ . That implies that

$$i_{\Sigma}^* dr = i_{\Sigma}^* d\bar{r}. \quad (21)$$

In view of (18),  $t + \int dr g^{-2}$  is a natural coordinate along  $\Sigma$  in the spacetime region  $N$ . Similarly  $\bar{t} + \int d\bar{r} \bar{g}^{-2}$  in  $\bar{N}$ . Therefore the forms  $dt + g^{-2} dr$  and  $d\bar{t} + \bar{g}^{-2} d\bar{r}$  may agree at  $\Sigma$  modulo suitable factors [null vectors have no natural normalization]. Indeed, setting  $g^2 dt + dr = \bar{g}^2 d\bar{t} + d\bar{r}$  at  $\Sigma$  is consistent with (21). Translating that in vielbein components we write

$$g(E^0 + E^1)|_{\Sigma} = \bar{g}(\bar{E}^0 + \bar{E}^1)|_{\Sigma}. \quad (22)$$

It is then not hard to show that condition (22) implies a Lorentz transformation  $\Lambda^a_{\bar{a}}$  which reads

$$\Lambda^a_{\bar{a}} = \begin{pmatrix} \frac{\bar{g}^2 + g^2}{2\bar{g}g} & \frac{\bar{g}^2 - g^2}{2\bar{g}g} \\ \frac{\bar{g}^2 - g^2}{2\bar{g}g} & \frac{\bar{g}^2 + g^2}{2\bar{g}g} \end{pmatrix}. \quad (23)$$

We suppress the trivial transformation of the angular components.

This transformation had to be explicitly known in order to express all tensor valued forms in the same basis: the fields  $\bar{\omega}_{\bar{b}}$  and  $\bar{\Omega}_{\bar{b}}$  must be transformed to obtain the components of  $\bar{\omega}$  and  $\bar{\Omega}$  in the directions of the basis  $E^a$ , which is the spacetime vielbein field appearing in equation (15).

Under a Lorentz transformation  $\Lambda_{\bar{a}}^a$  the spin connection  $\bar{\omega}$  transforms as

$$\bar{\omega}_{\bar{b}}^a = \Lambda_{\bar{a}}^a \bar{\omega}_{\bar{b}}^{\bar{a}} \Lambda_{\bar{b}}^{\bar{b}} + \Lambda_{\bar{b}}^a d\Lambda_{\bar{a}}^{\bar{b}}. \quad (24)$$

The transformation of its curvature tensor  $\bar{\Omega}$  follows and reads

$$\bar{\Omega}_{\bar{b}}^a = \Lambda_{\bar{a}}^a \bar{\Omega}_{\bar{b}}^{\bar{a}} \Lambda_{\bar{b}}^{\bar{b}}. \quad (25)$$

These transformations hold in the overlap of the support of  $E$  and  $\bar{E}$  in  $M$ . Under (20) and (24) the torsion tensor of  $\bar{E}$  transforms as  $d\bar{E}^a + \bar{\omega}_{\bar{b}}^a \bar{E}^{\bar{b}} = \Lambda_{\bar{a}}^a (d\bar{E}^{\bar{a}} + \bar{\omega}_{\bar{b}}^{\bar{a}} \bar{E}^{\bar{b}})$ . Being indeed a tensor, its components in both bases vanish simultaneously. Therefore  $\bar{\omega}_{\bar{b}}^a$ , given by (24), are indeed the components of the Levi-Civita connection  $\bar{\omega}$  in the basis which coincides with  $E^a$  at  $\Sigma$ . Everything needed in order to apply (15) has now been made explicit.

The quantities of importance are the pull-backs of the ‘jumps’  $(\bar{\omega} - \omega)_{\bar{b}}^a$  into  $\Sigma$ . Recalling (11) a straightforward calculation gives

$$\begin{aligned} i_{\Sigma}^* (\bar{\omega} - \omega)_1^0 &= 0, \\ i_{\Sigma}^* (\bar{\omega} - \omega)_1^i &= + \frac{\bar{g}^2 - g^2}{2g} \tilde{E}^i, \\ i_{\Sigma}^* (\bar{\omega} - \omega)_0^i &= - \frac{\bar{g}^2 - g^2}{2g} \tilde{E}^i. \end{aligned} \quad (26)$$

The jump of the angular components  $\omega_{\bar{b}}^i = \bar{\omega}_{\bar{b}}^i$  vanishes trivially. It is then easy to calculate the pull-back into  $\Sigma$  of the jump squared  $((\bar{\omega} - \omega)_{\bar{b}}^a)^2 = (\bar{\omega} - \omega)_c^a (\bar{\omega} - \omega)_c^b$ , which vanishes identically. [Such a simplification, due to the high symmetry of the configuration, was vaguely conjectured in Ref. [62].] Using (12) the forms  $(\bar{\Omega} + \Omega)_{\bar{b}}^a$  are also straightforwardly calculated.

Substituting everything in the matching condition (15) one finally finds

$$\begin{aligned} - \left\{ 3c_1 + c_2 \frac{2 - g^2 - \bar{g}^2}{r^2} \right\} \frac{\bar{g}^2 - g^2}{2gr} \times \\ \times i_{\Sigma}^* \epsilon_{01ijk} E^i E^j E^k \delta E^- = 0 \end{aligned} \quad (27)$$

where  $E^- := E^0 - E^1$ . Also  $E^+ := E^0 + E^1$ . The vielbein  $(E^{\pm}, E^i)$  is a basis adapted to  $\Sigma$ .

$E^-$  is the normal to  $\Sigma$  component of the vielbein in the sense that  $i_{\Sigma}^* E^- = 0$ , by (18). The presence of  $\delta E^-$  in (27) is a reminiscent of the tangent nature of the null normal vector. Explicitly the variation reads

$$i_{\Sigma}^* \delta E^- = \lambda(x)_+^- i_{\Sigma}^* E^+ + \lambda(x)_-^- i_{\Sigma}^* E^j, \quad (28)$$

for an arbitrary smooth field  $\lambda(x)_b^a$  with support on  $\Sigma$ . The component of the matching condition (27) involving  $\lambda(x)_+^-$  it is not trivial. Therefore (27) implies that

$$\left\{ 3c_1 + c_2 \frac{2 - g^2 - \bar{g}^2}{r^2} \right\} \frac{\bar{g}^2 - g^2}{2gr} = 0 \quad (29)$$

for all  $r$  along  $\Sigma$ . The solution  $\bar{g}^2 = g^2$  amounts to no discontinuity at all. The non-trivial solution is

$$3c_1 + c_2 \frac{2 - g^2 - \bar{g}^2}{r^2} = 0. \quad (30)$$

Recalling (14) we see that this equation can indeed be satisfied, and in fact  $g$  and  $\bar{g}$  must belong to *different branches* of the *same* Boulware-Deser solution given in (2). Thus we obtain the branch changing shock wave solution we claimed to exist in the beginning of this paper.

## V. COMMENTS

Mathematically, the spherical symmetric shock wave solutions of Lovelock gravity have been rather established. These shock waves are possible due to the multi-valued-ness of the spherically symmetric metrics of the theory. On the other hand, the multi-valued-ness itself is a symptom of a deeper pathology.

The whole thing can be better understood considering a simpler system. An elementary such system is a point particle with Lagrangian  $L = \mu v^2 + g v^4$ .  $v$  is the velocity of the particle and  $\mu$  and  $g$  are non-zero constants. The canonical momentum reads  $p = 2\mu v + 4g v^3$ . Varying the action  $\int dt L$  one obtains the Euler-Lagrange equation,  $\dot{p} = 0$ , the dot denoting time derivative. That is,  $p$  is constant in time. Let across some instant of time momentum change from  $p_{in}$  to  $p_{out}$  and velocity from  $v_{in}$  to  $v_{out}$ . The equation of motion requires that  $p_{in} = p_{out}$ . This translates to

$$(v_{in} - v_{out})(2\mu + 4g(v_{in}^2 + v_{in}v_{out} + v_{out}^2)) = 0. \quad (31)$$

This is a matching condition. Along with the solution  $v_{in} - v_{out} = 0$  i.e. velocity is constant, one obtains a solution in which velocity jumps according to the condition  $v_{in}^2 + v_{in}v_{out} + v_{out}^2 = -\mu/(2g)$ . The jumps may happen at arbitrary times. We learn that a source-free motion is not necessarily a uniform motion in this system. In fact,  $p = \text{constant}$  may correspond to an infinity of different piece-wise uniform motions.

From the Hamiltonian point of view, the problem arises in the following form. The Hamiltonian is a function of the canonical momentum  $p$ . Solving  $p = 2\mu v + 4g v^3$  for  $v$  the answer is multi-valued. Each one of the multiple solutions  $v(p)$  defines a different ‘canonical branch’ of the theory, as we may call it, with a different Hamiltonian  $H = p v(p) - L$ . We have seen that the source-free motion  $p = \text{constant}$  is neither simple nor unique in this theory, and in general involves

sudden changes of velocity according to the equation  $v_{in}^2 + v_{in}v_{out} + v_{out}^2 = -\mu/(2g)$ . These jumps are a sudden change of *canonical* branch as  $v_{in}$  and  $v_{out}$  are values of the velocity given by different  $v(p)$ . Thus the Hamiltonian by which the system evolves may change abruptly without a cause. A byproduct of this fact is that energy is not conserved in the free motion of this system. Moreover, one cannot uniquely relate the space of solutions to the space of the initial data i.e. there is no notion of classical phase space for this theory.

All that translates as it is in Lovelock gravity. The role of velocity is played by the extrinsic curvature which jumps across a hypersurface in vacuum. The Lagrangian is not linear in the curvature i.e. not quadratic in the extrinsic curvature. This leads, in the guise of many components and complicated formulas, to the same problems encountered in the simple point particle system. The multi-valued-ness of the Lovelock gravity Hamiltonian and some of its implications have been emphasized in Ref. [63]. In Ref. [37] it was shown by explicit examples that appropriate jumps of the extrinsic curvature across non-null hypersurfaces are allowed in vacuum, i.e. the canonical momentum does not change across the hypersurface nonetheless the theory changes canonical branch. Mathematically the results arise through factorized expressions analogous to equation (31): There is a continuity imposing factor which we require not to vanish, and another factor originating from the higher order terms which leads to a *soluble* equation. That was realized here in the form of equation (29) leading to equation (30) and our shock wave solutions. The factorization can be seen already at the level of equation (15) which is a complicate analogue of (31). In fact factoring out  $\bar{\omega} - \omega$  in (15) one obtains a rank three tensor whose vanishing is the general condition that must be satisfied by the fields in a change of canonical branch. The multiple canonical branches are due to the non-linearity of the Lagrangian in curvature, an outcome of which is the multi-valued-ness of solutions.

The shock waves presented here are spherically symmetric and exist in dimension higher than four. It takes the peculiar dynamics of Lovelock gravity for

them to arise. Now it is worth to mention that, even in Einstein gravity, once in higher dimensions spherical symmetry is not a necessary ingredient of Birkhoff's type of theorems. The reason why is that Israel's black hole uniqueness theorem [64][65], which says that static and asymptotically flat black holes are necessarily spherically symmetric, is not as strong in higher dimensions as it is in four: If one drops asymptotic flatness then the geometry of the 'angular' manifold [with the topology of the sphere] is not necessarily that of the round sphere, see the discussion in [66][67]. Einstein field equations restrict the 'angular' manifold geometry only that much. Now all that 'room to spare' might make possible even time-dependent solutions as it is numerically argued in Ref. [68] using an angular manifold with the topology of the three-sphere but not the geometry of the round sphere. In fact all that freedom is controlled once one switches on the Lovelock gravity terms available in those higher dimensions, as emphasized in the Refs. [69][36]. In this work we have seen that in Lovelock gravity  $C^0$  piecewise smooth time-dependent solutions exist in the form of shock waves even under spherical symmetry. The derivation goes through as it is with minor modifications in any other case of interest [one simply uses an 'angular' vielbein  $\tilde{E}^i$  with the appropriate properties] e.g. the Boulware-Deser-Cai metrics [70] or the metrics of Ref. [36] solutions of theory (1). Formulas become only more cumbersome, not essentially different, when including cubic or higher Lovelock terms in the appropriate dimension. The analogues of the shock waves presented here arise as long as the smooth solutions are multi-valued through elementary identities such as relation (14).

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